A FOUNDATION OF INFORMATION PROCESSES

In this paper the widely used concept of an information process is analyzed in more detail. Especially the information process is related to observable variables. The information process is usually understood to characterize investors’ expectations about future asset prices. We show under which conditions the information process is governed by a geometric Brownian motion and we give a rationale for stochastic volatility of the information process. The information process is also analyzed for the case of learning.

Introduction

Many studies in Financial Economics start from an exogenously given information process. This information process is assumed to characterize investors’ expectations. Most studies even assume that the information process is governed by a geometric Brownian motion without drift and constant volatility. While this is an elegant way of characterizing the information structure of an economy it suffers from the fact that investors’ expectations are not observable. Hence, it is virtually impossible to test the assumed stochastic process for investors' expectations empirically.

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2 See for example Brennan (1973), Brennan et. al. (2003), Franke et. al. (1999), Lüders and Peisl (2001) and Lüders and Franke (2004).
In this paper we use a simple model to relate the information process to observable information. The aim of this analysis is to provide a better understanding of the information process. What are sensible assumptions? How restrictive is the assumption of a constant volatility for the information process? Does stochastic volatility as proposed by Lüders and Peisl (2001) make sense? We also aim to relate two classes of theoretical asset pricing models, i.e. on the one hand the parsimonious models based on the information process and on the other hand more general models with production processes, as for example the model of Cox, Ingersoll and Ross (1985). The question how the information process is related to observable information is particularly important since it allows analyzing sensible assumptions on the information process as well as empirical tests of the information process. Recent studies have shown that the characteristics of the information process are very important for the characteristics of asset returns. Lüders and Peisl (2001), for example, have shown that stochastic volatility of the information process can explain certain so-called asset pricing anomalies. Recently researchers have analyzed the impact of learning, i.e. parameter uncertainty, on asset returns and have shown that this also helps explain certain asset pricing anomalies (e.g. Brennan and Xia, 2002; Johnson, 2002). We will show in this article that parameter uncertainty is basically an economic explanation for random volatility of the information process.

For the purpose of asset pricing all that is needed is a characterization of the pricing kernel and a characterization of the conditional distribution of future cash flows. Such a characterization of information is called an information structure. Technically the information structure is characterized by the filtration $G_t$. In every model explicitly or implicitly some assumptions about the information structure have to be made. How to model the relevant information, however, is somewhat arbitrary. In our model, where we make the assumption that no dividends are paid until the terminal date $T$, the relevant information is the (exogenous) distribution of the terminal value of the single company in the economy. This information can be modeled by the so-called information process. The information process characterizes the conditional expectations about the terminal value where the expectation is taken under the objective probability measure and with respect to the relevant information $G_t$. Consistent with the existing literature this “expectation process” will be called “information process”. Generally, the information process $I_t$ is defined as

$$I_t = E[X_T | G_t], \quad 0 \leq t \leq T,$$

where $X_T = I_T$ is the exogenous terminal value of the asset and the expectation is taken with respect to the objective probability measure. This process may be interpreted as the representative investor's conditional expectation about the terminal value of the asset (e.g. Brennan, 1973; Brennan et. al., 2003 and Franke et. al. 1999). Hence, in a risk neutral world the information process and the forward price of the asset coincide but generally the value of the information process exceeds the value of the forward price process by the value of the risk premium (see Franke et. al. 1999 and Lüders and Franke, 2004 for detailed analyses of the risk premium). Using an information process as defined in equation (1) has several advantages. First, it is a parsimonious way to characterize all the relevant information, i.e. it characterizes all the conditional distributions needed for pricing. Secondly, since it characterizes investors' expectations, it has an economic interpretation. However, the information process is not observable and therefore it is hard to decide which assumptions make sense. Besides a few obvious characteristics of the information process little is known. We only know that, since the information process characterizes conditional expectations, it is a martingale, i.e. its drift is zero. Intuitively, it is also clear that the expectations of a rational investor have to follow a martingale.
Otherwise, the investor could improve his forecasts by anticipating the expected change in his forecasts.

However, the information process may be derived from other observable information. For example, the information process could be derived from a complete general equilibrium model in which the investors observe some information flow (for example a news ticker) which is generated endogenously in the economy. To illustrate and to motivate the information process, in this paper we relate the information process to some exogenous information flow in the economy. In our model a stochastic process characterizing the book value of equity proxies the information flow in the economy. Since accounting data are available, linking the information process to the book value of equity process establishes a relation between the information process and observable data. This allows us to discuss sensible assumptions on the information process. This analysis is closely related to the recent research on learning (e.g. Brennan and Xia, 2002; Johnson, 2002; Veronesi, 2000 and Ziegler, 2002). While these articles derive asset prices for alternative information quality they do not explicitly derive the information process. This article focuses on the characteristics of the information process.

**Characterization of the Economy**

Let us assume that only one company exists and that the only source of information is the continually published book value of equity $b_v$. This number is the only information investors have about the terminal value of the company. Moreover, we assume that the company has no liabilities and that at the terminal date $T$ the book value of equity equals the exogenous terminal value of the company $X_T$. Thus, we assume that the market value and the book value are equal at the terminal date $T$. The reason for this assumption is that at the terminal date $T$ the company is liquidated and the investors receive $X_T$. This liquidation value is assumed to equal the book value at the terminal date $T$. In a fairly general form we write the book value process as

\[
b_v = b_v + \int_0^t \kappa(s) b_v ds + \int_0^t \gamma(s) b_v dW_s, \quad 0 \leq t \leq T,
\]

where the instantaneous drift $\kappa$ and the instantaneous volatility $\gamma$ are some general adapted stochastic processes. By choosing such a general representation for $\kappa$ and $\gamma$ we account for the fact that the book value of equity might be driven by several risk factors. This model nests more specific models for the book value of equity, including, for example, the geometric Brownian motion with constant instantaneous drift and constant instantaneous volatility.

Finding sensible and empirically relevant characterizations of the coefficients as well as discussing the general characteristics of the book value process is beyond the scope of this chapter. Instead of entering into this discussion we refer to the work of, for example, Ohlson (1992) and Ohlson (1995), Schwartz and Moon (2000) and Schwartz and Moon (2001). These studies, however, suggest that the coefficients are not constant, that investors face significant parameter uncertainty, that $\kappa$ might be mean-reverting and that the stochastic process might be driven by several risk factors. Therefore, in this article we will discuss the following cases. Firstly, we will assume that investors completely observe the realizations of the process $b_v$. For this case we consider first the example with $\kappa$ and $\gamma$ being constant and known. Since in this
case the book value process is governed by a geometric Brownian motion, we will consider it as a benchmark. Secondly, we will consider random coefficients. We will show in particular that an information process with stochastic volatility can be derived from a book value process with stochastic volatility. Afterwards we will analyze an alternative level of observability. We will discuss the case, when the process \( b_v \) is still completely observable but the investors do not know the instantaneous drift \( \kappa \). In this case investors have to estimate the parameters from historical data. This analysis provides also insights into the effects generated by learning.

**Complete Information and Constant Coefficients of the Book Value Process**

We analyze first the case with complete observation and constant coefficients. Therefore, we assume a probability space \((\Omega, G, P)\) and a standard Brownian motion \(W\) on \((\Omega, G, P)\).

Assume that the book value process \( b_v \) is governed by the following stochastic differential equation

\[
 b_v_t = b_v_0 + \int_0^t \kappa b_v_s \, ds + \int_0^t \gamma b_v_s \, dW_s , \quad 0 \leq t \leq T, \tag{3}
\]

with \( \kappa \) and \( \gamma \) being positive constants. Thus, this case corresponds to an ideal world for an investor since it implies that he has complete knowledge about the company and its environment, although the outcomes are subject to random shocks. Let us derive the information process \( I \).

Technically we assume that investors' information is characterized by the filtration generated by the process \( b_v \), \( \Psi = \{ \Psi_t := \mathcal{F}(b_v; 0 \leq s \leq t) \}_{t \in [0,T]} \) and completed by all the \( P \)-null sets. \( \Psi \) is the filtration generated by the book value process. Thus, economically \( \Psi_t \) characterizes what investors have learned by observing the book value process between time 0 and time \( t \). We apply the Theorem of Feynman-Kac to get a characterization of the information process.

**Proposition 1**

If a) the book value process \( b_v \) is governed by equation (3) and b) the information available to the investor is characterized by the filtration \( \Psi = \{ \Psi_t := \mathcal{F}(b_v; 0 \leq s \leq t) \}_{t \in [0,T]} \) generated by the process \( b_v \) and completed by all the \( P \)-null sets, then the information process follows a geometric Brownian motion with drift zero and its instantaneous volatility is equal to the instantaneous volatility \( \gamma \) of the book value process \( b_v \).

**Proof**

By the Theorem of Feynman-Kac there exists a unique function \( v : [0,T] \times \mathbb{R} \to \mathbb{R} \) satisfying

\[
 \frac{\partial v(t,x)}{\partial t} + \frac{\partial v(t,x)}{\partial x} \kappa x + \frac{1}{2} \frac{\partial^2 v(t,x)}{\partial x^2} \gamma x^2 = 0 \tag{4}
\]

\[
 v(T,x) = x
\]
and the information process \( I \), defined by \( I_t = E \left[ b v_t | \Psi_t \right] \) for \( 0 \leq t \leq T \), can be characterized by \( I_t = v(t, b v_t) \) for \( 0 \leq t \leq T \). Solving the partial differential equation yields:

\[
v(t, x) = x \exp \left( \kappa (T - t) \right).
\]

Applying Itô's Lemma yields \( dI_t = I_t \gamma dW_t \), \( 0 \leq t \leq T \).

q.e.d.

This proposition sheds some light on the characteristics of the information process. Given the book value process is a geometric Brownian motion, the investor completely observes the process and he knows the values of the coefficients, then the information process is also governed by a geometric Brownian motion. Moreover, the instantaneous volatilities of both processes are the same. This result is also intuitively clear. Taking expectations, the investor anticipates the drift of the book value process. But the uncertainty, the pure noise of the diffusion term, cannot be reduced. There is no additional information he may extract to reduce the uncertainty about the terminal outcome. Hence, his expectations are governed by a stochastic process with the same instantaneous volatility as the book value process. Finally, the above example has shown a framework which leads to an information process governed by a geometric Brownian motion. Since an asset price process governed by a geometric Brownian motion implies that the information process is also governed by a geometric Brownian motion, this information structure is consistent with the Black-Scholes model (Franke et. al. 1999; Lüders and Peisl, 2001). The important assumption in this model is that the investor completely observes the book value process \( b v \) and that the coefficients are known. However, in reality one of the investors' main problems is to quantify the true drift of some book value or earnings process. Moreover, in reality we may not expect drift and volatility to be constant. It seems more realistic to assume these coefficients to be stochastic (for empirical evidence, Schwartz and Moon, 2000; Schwartz and Moon, 2001).

We will now briefly discuss information processes when coefficients are random. Therefore we will discuss two special cases: \( i \) random drift which depends on the level of the book value and \( ii \) stochastic volatility of the book value process. Later on, to understand the impact of parameter uncertainty, we will assume that the drift of the book value process is not known.

**Complete Information and Random Coefficients of the Book Value Process**

**Random Drift of the Book Value Process**

The geometric Brownian motion as a model to describe the evolution of the book value of equity does not capture the characteristics of the behaviour of book values of equity in reality. One straightforward extension would be to replace the constant drift by some random drift. For example, due to business cycles one might expect the drift to fluctuate around a long-term mean level. Such time-varying drift of earnings or book value processes is prevalent in the modeling of
such processes. To get an understanding of the impact for the information process we will therefore assume here that

\[
bv_t = bv_0 + \int_0^t \kappa(s, bv_s) bv_s ds + \int_0^t \gamma bv_s dWs \quad , \quad 0 \leq t \leq T ,
\]

\[
bv_0 > 0 ,
\]

with the volatility \( \gamma \) being some positive constant and the drift \( \kappa(t, bv_t) \) being a deterministic function of \( bv_t \) and \( t \). The other assumptions of the model, as for example the one risk factor structure, are unchanged. Analogously to the case with constant coefficients, \( I_t \) is then characterized by a deterministic function \( I : [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \) satisfying

\[
\frac{\partial v(t, x)}{\partial t} + \kappa(t, x) x + \frac{1}{2} \frac{\partial^2 v(t, x)}{\partial x^2} \gamma^2 x^2 = 0
\]

\[
v(T, x) = x .
\]

Hence, technically few things change. Only the constant drift coefficient is replaced by a deterministic function. Even if we cannot solve this new partial differential equation analytically we may, however, derive some properties of the information process. Note that the instantaneous volatility of the information process \( \sigma(\cdot) \) is a deterministic function characterized by

\[
\sigma(\cdot) = \frac{\partial v(t, x)}{\partial x} \frac{x}{v(t, x) \gamma} .
\]

Equation (7) implies that if the instantaneous volatility of the information process is constant, then the elasticity of \( v(t, x) \) with respect to \( x \) must be constant, too. However, solving equation (6) usually yields a random volatility \( \sigma(\cdot) \). Moreover, for the instantaneous volatility of the information process to equal the instantaneous volatility of the book value process the elasticity has to be equal to 1. Combining equation (7) with equation (6) we find that constant volatility of the information process implies that the drift of the book value process must be a function of time \( t \) only. Hence, in general the instantaneous volatility of the information process will be random if we assume that the drift of the book value process is random.

**Stochastic Volatility of the Book Value Process**

Another possible extension would be to assume that the book value of equity is driven by a second risk factor. Prevalent in this class of stochastic processes are stochastic volatility models. Let us therefore consider the following model: While we still assume a finite time horizon \( T \), we assume two uncorrelated Brownian motions \( W \) and \( B \) on \( (\Omega, G, P) \). The book value of equity shall now admit the following characterization

\[
bv_t = bv_0 + \int_0^t k bv_s ds + \int_0^t \gamma bv_s dWs \quad , \quad 0 \leq t \leq T ,
\]

\[
bv_0 > 0 ,
\]

3 See, for example, Schwartz and Moon (2000) and Schwartz and Moon (2001).
with the instantaneous volatility $\gamma_t$ being governed by

$$\gamma_t = \gamma_0 + \int_0^t \zeta ds + \int_0^t dB_s, \quad 0 \leq t \leq T, \quad (9)$$

For simplicity we assume $\kappa$, $\varsigma$, and $\iota$ to be positive constants. It follows from the Theorem of Feynman-Kac that the partial differential equation characterizing the information process is given by

$$0 = \frac{\partial}{\partial t} v(t,x,y) + \frac{\partial}{\partial x} v(t,x,y) \kappa x + \frac{\partial}{\partial y} v(t,x,y) \varsigma + \frac{\partial^2}{\partial x^2} v(t,x,y) x^2 + \frac{\partial^2}{\partial y^2} v(t,x,y) y^2$$

$$+ \frac{1}{2} \frac{\partial^2}{\partial x^2} v(t,x,y) x^2 y^2 + \frac{1}{2} \frac{\partial^2}{\partial y^2} v(t,x,y) y^2$$

$$v(T,x,y) = 0. \quad (10)$$

Solving the partial differential equation (10) and applying Itô's Lemma yields the following information process

$$dI_t = \gamma_t dW_t, \quad 0 \leq t \leq T,$$

$$d\gamma_t = \varsigma dt + \iota dB_t, \quad 0 \leq t \leq T,$$

$$I_T = b v_T. \quad (11)$$

Hence, similar to the case with constant volatility and constant drift, the information process has characteristics which are very similar to the book value process: The instantaneous volatility is the same, but the drift is removed. This information process is very similar to the information process assumed in Lüders and Peisl (2001). Hence, their information process basically characterizes an economy where the book value of equity is governed by a two risk factors. Let us now turn to the question how the information process is affected if we assume that investors have to estimate the drift of the book value process from past observations instead of knowing it exogenously. This question is closely related to the recent research on asset pricing with learning (e.g. Brennan and Xia, 2002).

**Unknown Drift of the Book Value Process**

In the following we assume that the book value is governed by

$$b v_t = b v_0 + \int_0^t \tilde{\kappa} b v_s ds + \int_0^t \gamma b v_s dW_s, \quad 0 \leq t \leq T, \quad (12)$$

$$b v_0 > 0,$$

with $\overline{\gamma}$ some positive constant and $\tilde{\kappa}$ is a (time-independent) one-dimensional, real-valued random variable. $W$ is a one-dimensional, standard Brownian motion on $\Omega, G, P$. Technically we will again analyze

$$I_t = E\left(b v_T | \Psi_t\right), \quad 0 \leq t \leq T, \quad (13)$$
where $\Psi = \{\Psi_t := \zeta(bv_s: 0 \leq s \leq t)\}_{t \in [0, T]}$ is the filtration generated by the observation of $bv$ (and completed by all the $P$-null sets). The difference between this model and the model discussed earlier is that now the investor does not know the true drift. Therefore, the drift $\tilde{\kappa}$ is assumed to be a random variable and the investor has to estimate the realization of this random variable. Let us now turn to the estimation of the instantaneous drift $\hat{\kappa}$. First, define the investor's estimate $\hat{\kappa}$ of the instantaneous drift $\kappa$:

$$\hat{\kappa}_t = E(\kappa_t | \Psi_t), \quad 0 \leq t \leq T.$$  

Since by assumption the investor does not know $\kappa$ but observes the book value process $bv$, the best he can do is to estimate the instantaneous drift $\hat{\kappa}$ on the basis of the information generated by the book value process $bv$ and his prior beliefs which are, by assumption: $\kappa$ normally distributed with mean $E(\kappa)$ and variance $a^2 = E(\kappa^2) - (E(\kappa))^2$. Therefore, in equation (14) the investor's estimate $\hat{\kappa}$ is defined to be the conditional expectation of $\kappa$ where the expectation is conditioned on the filtration $\Psi$ which represents the information the investor gets by observing the book value process $bv$. Applying the so-called Kalman-Bucy-filter (see Liptser and Shiryaev, 1977, and for an application Ziegler, 2002) yields that the estimate $\hat{\kappa}$ is characterized by

$$\hat{\kappa}_t = E(\kappa_t) + \int_0^t \frac{a^2}{\gamma^2 + a^2 s} \gamma dW_s, \quad 0 \leq t \leq T.$$  

Since the true drift is not known, the investor also has to estimate the Brownian motion. The estimated Brownian motion $\hat{W}_t = E(W_t | \Psi_t)$ is governed by (see Liptser and Shiryaev, 1977)

$$d\hat{W}_t = dW_t + \frac{\kappa - \hat{\kappa}_t}{\gamma} dt$$

$$= \frac{1}{\gamma} (\gamma dW_t + (\kappa - \hat{\kappa}_t) dt)$$

$$= \frac{1}{\gamma} (\kappa dt + \gamma dW_t - \hat{\kappa}_t dt), \quad 0 \leq t \leq T.$$  

Hence, the estimated Brownian motion is equal to the deviation of $\frac{dbv_t}{bv_t}$ from $\hat{\kappa}_t dt$ per unit volatility $\gamma$. It follows from equation (15) that the estimated drift $\hat{\kappa}_t$ is updated according to

$$d\hat{\kappa}_t = \frac{a^2}{\gamma^2 + a^2 t} \gamma d\hat{W}_t, \quad 0 \leq t \leq T.$$  

Ziegler (2002) shows that inserting (16) into (17) yields a nice intuition for the updating process. The estimated drift is updated by the deviation of $\frac{dbv_t}{bv_t}$ from $\hat{\kappa}_t dt$ weighted by $\frac{a^2}{\gamma^2 + a^2 t}$. This factor can be interpreted as a measure of uncertainty about $\hat{\kappa}_t$.

Now, let us turn to our original question: How does parameter uncertainty influence the information process? In order to derive the information process $I$, we first rewrite our model with incomplete information as a model with complete information:
\[ dbv_t = b v_t \left( \hat{k}_t dt + \gamma d\hat{W}_t \right) \quad , 0 \leq t \leq T, \]

and \( \hat{k}_t \) is defined by equation (15). Thus, the incomplete information model is transformed into a complete information model where \( \hat{k} \) is replaced by \( \hat{k} \) and \( W \) is replaced by \( \hat{W} \). According to the Theorem of Feynman-Kac \( I_t \) can be characterized by a function \( \alpha : [0,T] \times \mathbb{R}^2 \rightarrow \mathbb{R}, (t,x_1,x_2) \mapsto \alpha(t,x_1,x_2) \) with

\[ 0 = \frac{\partial \alpha(t,x_1,x_2)}{\partial t} + \frac{\partial \alpha(t,x_1,x_2)}{\partial x_1} x_1, x_2 \]

\[ + \frac{1}{2} \left( \frac{\gamma^2}{\sigma^2} \alpha \left( \frac{\partial^2 \alpha(t,x_1,x_2)}{\partial x_1^2} + 2 \frac{a^2}{\gamma^2 + a^2 t} \frac{\partial^2 \alpha(t,x_1,x_2)}{\partial x_1 \partial x_2} \right) \right) \]

\[ \alpha(T,x_1,x_2) = x_1 \]

with \( I_t = \alpha(t,bv_t,\hat{k}_t) \) and hence the information process \( I \) is governed by the following stochastic differential equation

\[ dI_t = \frac{\partial \alpha(t,bv_t,\hat{k}_t)}{\partial bv_t} bv_t d\hat{W}_t + \frac{\partial \alpha(t,bv_t,\hat{k}_t)}{\partial \hat{k}_t} \gamma d\hat{W}_t \]

\[ 0 \leq t \leq T, \]

\[ I_0 > 0. \quad (20) \]

Writing \( I \) in the usual notation as

\[ dI_t = \sigma(t,I_t) I_t d\hat{W}_t \quad , 0 \leq t \leq T, \]

\[ I_0 > 0. \quad (21) \]

With an appropriate \( \sigma \), it follows from equation (20), equation (21) and \( I_t = \alpha(t,bv_t,\hat{k}_t) \) that the information process would have constant instantaneous volatility \( \sigma \) if and only if

\[ \frac{\gamma}{\alpha(t,x_1,x_2)} \left( \frac{\partial \alpha(t,x_1,x_2)}{\partial x_1} bv_t + \frac{a^2}{\gamma^2 + a^2 t} \frac{\partial \alpha(t,x_1,x_2)}{\partial x_2} \right) = \text{constant}. \quad (22) \]

It can be checked by combining equation (19) and condition (21) that the instantaneous volatility is not constant. Thus, the instantaneous volatility of the information process \( I \) becomes random because of parameter uncertainty. Hence, parameter uncertainty provides also an economic, respectively filter theoretic, foundation for an information process with instantaneous volatility being a deterministic function of \( bv_t \) and time \( t \).
Conclusion

Many models use the information process (i.e. the representative investor’s expectations) as a parsimonious way to characterize the information structure in the economy. But since the information process is not observable it is hard to decide which characteristics of the information process make sense. In this paper we use a simple model to relate the information process to observable information. In our case observable information on a company is proxied by the book value of equity. We derive the information process for alternative assumptions on the book value process. We find that unless the book value process is assumed to be governed by a geometric Brownian motion with constant coefficients, information processes have random volatility. Hence, our analysis suggests the use information processes with random volatility.

References


